11

Frege's Relation to Dedekind: *Basic Laws* and Beyond

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11.1 INTRODUCTION

Among all of Gottlob Frege's contemporaries, the thinker who is probably closest to him in terms of their basic projects is Richard Dedekind. Like Frege, Dedekind attempted to reconstruct the theories of the natural and real numbers in purely logical terms. As in Frege's case, this involved using a general theory of classes, relations, and functions. More specifically, both used infinite classes in their definitions of numbers, thus proposing to "ground the finite on the infinite" (as Hilbert would later put it).[1] Their approaches were closely related in terms of important details too, such as their analyses of mathematical induction. Frege and Dedekind were equally shocked about Russell's antinomy initially, because it undermines their projects in parallel ways.[2] And finally, several of these similarities may well be explained by the fact that both were educated, as mathematicians, at the University of Göttingen, although their student years there didn't overlap.[3]

Of course, there are also significant differences between Frege's and Dedekind's perspectives. It has often been noted that, while Frege highlights proof-theoretic aspects in his work on logic and the foundations of mathematics

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[1] Hilbert found this proposal "dazzling and captivating" but problematic in the end; cf. Hilbert (1922, paragraph 21). For further discussion, cf. Ferreirós (1999, 254) and Ferreirós (2009).

[2] Frege's strong reaction to Russell's antinomy is well known. After being told about closely related antinomies, probably by Georg Cantor, Dedekind reportedly developed doubts about whether "human thought is fully rational" (Ewald, 1996, vol. 2, 836). On the other hand, in the Preface to the third edition of *Was sind und was sollen die Zahlen?*, from 1911, he expressed confidence again that these problems could be overcome; cf. Dedekind (1932, vol. 3, 343).

[3] For Dedekind's Göttingen background, cf. Ferreirós (1999, chs. 1–2); for Frege, cf. Tappenden (2006). For more on the other similarities, cf. the articles by me listed in the References.

(by introducing a formal language, his "Begriffsschrift", and a corresponding purely syntactic deduction system), Dedekind's approach is more model-theoretic (with his investigation of questions involving models, homomorphisms, and categoricity). Similarly, it is often said that Frege conceives of the natural numbers essentially as cardinal numbers, whereas Dedekind makes the ordinal conception of the natural numbers fundamental. Moreover, Dedekind proposes a structuralist account of the nature of mathematical objects, while for Frege they have "intrinsic" properties, not just the "relational" ones characteristic of structuralism. Even more on the philosophical side, Frege is usually taken to endorse a "platonist" position, while Dedekind tends to be criticized for holding "psychologistic" views.[4]

Given these similarities and significant contrasts, it is surprising that Frege and Dedekind did not interact more. They were near contemporaries: Dedekind was born in 1831 and died in 1916, Frege lived from 1848 to 1925. Frege was thus seventeen years younger than Dedekind. But they do not seem to have corresponded in any substantive way. (Neither Frege's nor Dedekind's *Nachlass* contains any corresponding letters.) They also did not review each other's works in print. (Unlike for, say, Cantor's, Husserl's, and Schröder's works, Frege did not publish separate reviews of any of Dedekind's writings. As far as I am aware, Dedekind did not publish any such reviews.[5]) Finally, they seem to have developed their central ideas in the foundation of mathematics independently (or largely so, compare below). One of the noteworthy aspects of Frege's *magnum opus*, *Basic Laws of Arithmetic*, is, then, that this is the work in which he comments on Dedekind's writings in most detail.

In this essay, I want to reconsider Frege's relation to Dedekind, while also putting it into a broader context. I will start with some background information. In §11.2, a quick reminder about Dedekind's foundational contributions will be provided; in §11.3, I will consider, very briefly, Frege's and Dedekind's respective receptions by other writers, including several "Frege-inspired" criticisms of Dedekind that are widespread in the analytic tradition.[6] Against that background, we will turn to their explicit remarks about each other. §11.4 will contain a survey of Frege's relevant comments; in §11.5, I will bring up some remarks by Dedekind in turn. In §§11.6 and 11.7, Frege's criticisms of Dedekind from *Basic Laws* will be analyzed in more detail, after dividing them into more minor criticisms and the major, lasting ones. In §11.8, finally, I will suggest a way in which Frege's and Dedekind's approaches can be brought into more fruitful contact, thus clarifying that my goal is not to take sides but to mediate between the two.

[4]For Frege's "platonism", cf. Reck (2005a,b); for Dedekind's structuralism and his alleged "psychologism", cf. Reck (2003), also Yap (2017). For their receptions, cf. Reck (2013).

[5]For Frege's published reviews of other writers, cf. Frege (1984). I will discuss other forms in which Frege and Dedekind commented on each other's works in what follows.

[6]For more on the broader reception of both Frege and Dedekind, cf. again Reck (2013). The present essay complements the latter by focusing more on Frege's own reaction to Dedekind.

11.2 A BRIEF SUMMARY OF DEDEKIND'S FOUNDATIONAL CONTRIBUTIONS

Almost all of Dedekind's contributions to the foundations of mathematics are contained in two small booklets: *Continuity and Irrational Numbers* (1872) and *The Nature and Meaning of Numbers* (1888).[7] In the former, he uses what are now called 'Dedekind cuts' to provide a systematic account not only for the irrational, but for all real numbers. In doing so, he starts from the system of rational numbers (implicitly seen as constructible, in two steps, out of the natural numbers and the integers as equivalence classes of pairs).[8] Dedekind points out that, while the rational number system is dense, it is not line-complete, i.e., it contains "gaps" such as that corresponding to $\sqrt{2}$. His cuts are a way of identifying these gaps "purely arithmetically". With respect to all the cuts, he then introduces—by a process of "abstraction" and "free creation"—corresponding real numbers. The result is a number system that is a complete ordered field, and an approach that allows for rigorous proofs of theorems central to the Calculus (concerning operations on square roots and the limits of increasing bounded sequences, among others).

Already in his 1872 essay, Dedekind uses set-theoretic constructions at crucial points. In his 1888 essay, he reflects on this procedure more explicitly and systematically. He also provides a novel, purely "logical" account of the natural numbers. Starting with the notions of object ["*Ding*"], set ["*System*"], and function ["*Abbildung*"]—all taken to be part of "logic"—his central definitions are those of "infinity" (being Dedekind-infinite), "chain" (a set closed under a given function), and "simple infinity" (the closure of a singleton set in an infinite set under an injective function). Famously, Dedekind establishes that any two simple infinities are isomorphic. He also provides justifications for definitions by recursion and proofs by mathematical induction. More controversially, he "proves" the existence of an infinite set, and, hence, of a simply infinite set (Theorem 66). Given the existence and unique (categorical) characterization of simple infinities, "abstraction" and "free creation" are used, once again, to introduce the natural numbers, thus conceiving of them purely structurally and as finite ordinal numbers. Dedekind shows how to define addition and multiplication recursively, and, finally, how initial segments of his number system can serve to measure the cardinality of finite sets.[9]

Many of Dedekind's contributions, as just summarized, have become accepted, indeed canonical parts of logic, set theory, and the foundations of mathematics. But since philosophers tend to zero in on the controversial aspects right away—and since I will try to defend Dedekind even in connection with those—let me add some details concerning the two most infamous parts

[7] For fuller references, concerning both Dedekind and Frege, see the References. As the timeline matters, I will refer to all of their works by their original publication dates.

[8] For Dedekind's views on the integers and the rational numbers, cf. Sieg and Schlimm (2005).

[9] For further details concerning both of Dedekind's essays, cf. Reck (2003, 2008).

of *The Nature and Meaning of Numbers*. First to "Theorem 66", where Dedekind means to prove the existence of an infinite set. For this purpose, he starts with "my own realm of thoughts, i.e., the totality S of all things which can be an objects of my thought"; as a function f on that "system", he introduces the mapping of any element s to "the thought s' that s can be object of my thought"; and as a distinguished element of S, different from all the values of f, he points towards "my own self" (Dedekind, 1963, 64, translation modified slightly). The suggestion is, then, that the chain over the singleton {Dedekind's self} generated by f in S is infinite, indeed simply infinite.

The second controversial part of Dedekind's essay is his "Definition 73". In it, he introduces the natural numbers as follows:

> If in the consideration of a simply infinite system N set in order by a function φ we entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another in which they are placed by the order-setting function φ, then are these elements called *natural numbers* or *ordinal numbers* or simply *numbers* … With reference of this freeing the elements from every other content (abstraction) we are justified in calling numbers a free creation of the human mind.
>
> (Dedekind, 1963, 68, emphasis in the original, translation slightly modified)

I will return to both of these aspects—Dedekind's "existence proof" for infinite sets, as well as the use of "abstraction" and "free creation" that grounds his structuralist conception of number—several times as we go along.

11.3 COMPARING FREGE'S AND DEDEKIND'S RECEPTIONS

Clearly, Dedekind's two foundational booklets correspond closely to Frege's main works.[10] One difference not highlighted so far is that, while Dedekind treats the real numbers first (in 1872) and the natural numbers second (in 1888), the order is reversed in Frege's writings (in volumes I and II of *Basic Laws*, published in 1893 and 1903, respectively). A further similarity is that both Frege's and Dedekind's foundational works were widely ignored or dismissed initially, both by philosophers and mathematicians (with some exceptions). Frege's lament about that fact, in volume I of *Basic Laws*, is well known. (It includes a complaint that Dedekind did not pay sufficient attention to his writings; more on that below.) But Dedekind's works on the natural and real numbers were not greeted with immediate, general enthusiasm either, at least relative to the canonical status they acquired later, since they were taken to be too abstract and basically pointless.[11] In retrospect, both thinkers were far ahead of their time, in related but slightly different ways.

In Frege's case, there was the more sympathetic reception by Russell, Wittgenstein, Carnap, and Husserl, to be sure. Yet it took until the 1950s for his

[10] In this section I draw heavily on Reck (2013), which also contains further references.

[11] Cf. the brief discussion of Dedekind's reception in Sieg and Schlimm (2005, 121).

writings to have a more general impact in philosophy. Then again, from the 1960s and 70s on, at least since Michael Dummett's important commentaries, Frege has played a central role in analytic philosophy, especially with his views on logic and language. His philosophy of mathematics started to be reconsidered seriously again in the 1980s, by Crispin Wright and others. Within the discipline of mathematics, in contrast, Frege's recognition has remained rather limited, at least outside of mathematical logic. Dedekind's reception took a significantly different course. His more general mathematical works, in algebra and algebraic number theory, had a huge impact in mathematics already in the nineteenth century. His foundational writings, while at first not appreciated very widely in mathematics, had a few early champions too, such as Ernst Schröder in his *Vorlesungen über die Algebra der Logik* (Schröder, 1890–1905). And after their assimilation by Hilbert, Zermelo, and others, Dedekind's technical contributions to the foundations of mathematics became an integral part of mathematical logic, especially of axiomatic set theory.

Dedekind's reception within the discipline of philosophy is most striking, however. It is not that philosophers paid no attention at all. Russell, for one, read his writings early (like Cantor's, Peano's, and Frege's).[12] He also praised Dedekind for his "brilliant contributions", including his general theory of relations, the notion of "progression" (Russell's term for a simple infinity), his analysis of mathematical induction, his definition of the notion of infinity, and his use of cuts in connection with the reals. But also from early on (beginning with *The Principles of Mathematics*, 1903), Russell criticized the more philosophical aspects of Dedekind's views. Thus, he characterized Dedekind's ordinal conception of numbers as more "complicated" than the cardinal conception proposed by Frege and himself; he observed that Dedekind's Theorem 66 appears to import non-mathematical, and non-logical, considerations into the foundations of mathematics; and by insisting that numbers have to be "intrinsically something", he voiced puzzlement about Dedekind's notion of "abstraction" and his structuralist conception of mathematical objects. Finally, there is Russell's famous quip about "postulation" having the advantages of "theft over honest toil", which is often taken to apply to Dedekind (but see below).

Russell's criticisms of Dedekind produced many echoes later on. Let me mention only two, but very prominent ones. First, in an influential discussion of neo-Fregean ideas in the philosophy of mathematics, George Boolos comments on Dedekind's Theorem 66. In a tone considerably more polemical than Russell's, he dismisses it as "one of strangest pieces of argumentation in the history of logic" (Boolos, 1998, 202). Second, in Michael Dummett's widely read book, *Frege: Philosophy of Mathematics* (1991), the following can be found: Like Russell, Dummett criticizes Dedekind's Theorem 66 as illegitimately importing non-mathematical considerations into the foundations

[12]Husserl discussed both Frege and Dedekind even earlier; cf. his *Philosophy of Arithmetic* (1891). But like in Russell's case, Frege receives more attention by him than Dedekind.

of mathematics; again like Russell, he favors a cardinal conception of numbers over an ordinal conception. Dummett picks up on Russell's argument against structuralist views too, and more specifically, dismisses positions like Dedekind's as "mystical structuralism". With respect to the idea—supposedly shared by Husserl, Cantor, and Dedekind—that the mind can "create" mathematical objects, he declares: "Frege devoted a lengthy section of *Grundlagen*, §§29–44, to a detailed and conclusive critique of this misbegotten theory" (Dummett, 1991, 50). Finally, Dummett bolsters this polemic against Dedekind by calling Frege "the greatest philosopher of mathematics yet to have written" (1991, 321). Overall, both Boolos and Dummett appear to think that being "pro-Frege" entails being strongly "anti-Dedekind".

In contrast to such polemical and dismissive responses to Dedekind by various philosophers, from Russell to some of his recent followers, there has been a revival of structuralist views in the philosophy of mathematics over the last few decades (brought about by Paul Benacerraf, Michael Resnik, Stewart Shapiro, Geoffrey Hellman, and Charles Parsons, among others). In this context, Dedekind is often viewed, or appropriated, as a distinguished ancestor.[13] However, even in the corresponding defenses of various versions of structuralism ("eliminative" and "non-eliminative"), some of the Russell- or Frege-inspired criticisms of Dedekind just mentioned come up again. In particular, Dedekind's appeal to "abstraction" and "free creation" continues to be seen, explicitly or implicitly, as constituting a problematic form of psychologism, one of which Frege helped to rid us.[14] But speaking about Frege, let me now turn to his own response to Dedekind.

11.4 AN OVERVIEW OF FREGE'S COMMENTS ON DEDEKIND

As already mentioned, most of Frege's explicit comments on Dedekind occur in *Basic Laws of Arithmetic*. It should help to start with a quick overview of them, before providing a more detailed analysis later on. In volume I of *Basic Laws* (1893), Dedekind is mentioned in the Preface (vii–viii, xi) and in the Introduction (1–3). At both places, Frege considers Dedekind's basic framework in *The Nature and Meaning of Numbers*, and especially, his views about "systems". At issue is, thus, the general project of reducing arithmetic to logic, with the focus on the form a foundational system should take (more so than, say, details of Dedekind's treatment of the natural numbers). In volume II of *Basic Laws* (1903), where Frege himself turns to the real numbers, it is Dedekind's views about these numbers, as introduced in *Continuity and Irrational Numbers*, that come under scrutiny. This happens within a more general critique of views about the reals—held by Georg Cantor, Hermann Hankel,

[13] For references and relevant remarks, cf. Reck (2003, 2013), also Reck and Price (2000).

[14] Outside the analytic tradition, there are exceptions; cf. Ernst Cassirer, as discussed in Reck (2013) and Yap (2017). Within analytic philosophy, cf. Tait (1997) and Reck (2003).

Otto Stolz, and others—that Frege finds in the literature (Frege, 1903, 140–9). Finally, Frege returns to Dedekind briefly in the Afterword to volume II, in connection with Russell's antinomy (253).

It should be emphasized, especially for my purposes, that Frege does not always criticize Dedekind at those places. In fact, volume I of *Basic Laws* starts with some rather positive comments. In its Preface, Frege calls Dedekind's 1888 essay "the most thorough study I have seen in recent times concerning the foundations of arithmetic" (vii).[15] He also notes that "Mr. Dedekind too is of the opinion that the theory of numbers is a part of logic" (viii), i.e., he acknowledges him to be a fellow logicist. And when turning to Dedekind's views on "systems" in the Introduction, Frege praises him for adopting an extensional view about them (1–2). In volume II of *Basic Laws*, further praise, or at least acknowledgements of more affinities, can be found. Frege commends Dedekind for adopting an anti-formalist view, in the sense of distinguishing numbers explicitly from corresponding numerals. He also observes that Dedekind, like himself, treats equality in arithmetic as (objectual) identity (Frege, 1903, 140). Finally, in the Afterword, where Frege grapples with the challenge posed by Russell's antinomy, he mentions Dedekind as "being a companion in my misery" as caused by it (253).

Of course, Frege wouldn't be Frege if he left it at such positive comments. But before considering his various criticisms in more detail, let me list the other places in Frege's writings where Dedekind comes up, positively or negatively, including some letters and posthumously published pieces. I have found five such places.

Frege's first mentioning of Dedekind, as far as I am aware, occurs in a letter to one Walter Brix (the author of a dissertation on the foundations of arithmetic), from late 1890 or early 1891 (precise date unknown).[16] In it, he declares: "I am playing with the idea of surveying critically, and illuminating comparatively, the views of Helmholtz, Kronecker, and, especially, Dedekind and others on number" (Frege, 1969, 12, my translation).[17] In a letter to Peano, later in 1891 (precise date again unknown), Frege admonishes various writers, including Dedekind, for not distinguishing sharply between the element and the subset relation (Frege, 1980, 109). Third, in his 1892 review of Cantor's essay, *Zur Lehre vom Transfiniten*, both Dedekind's definition of infinity and his theory of "chains" are mentioned positively, in relation to Frege's own ideas (Frege, 1984, 180). In a posthumous piece from 1897, called 'Logic', Frege comments, approvingly again, on the notion of "thought"

[15] While my page references will be to Frege (1893) and Frege (1903), all quotations will be taken from Frege (2013). As Frege's original pagination is preserved in it, it is easy to go back and forth.

[16] This letter is contained in Frege (1969, 12), but not in its English translation, Frege (1980).

[17] In the original German: "Ich trage mich mit dem Gedanken, einen kritischen Streifzug zu unternehmen und dabei die Ansichten von Helmholtz, Kronecker und besonders Dedekind und Anderen über die Zahl vergleichend zu beleuchten" (Frege, 1969, 12).

in Dedekind's Theorem 66; in fact, he sees Dedekind as using an objective notion of "thought" similar to his own (Frege, 1997, 236–7). Finally, in his 1899 parody, 'On Mr. H. Schubert's Numbers', Dedekind's understanding of equality as identity is contrasted favorably with Schubert's muddled views (Frege, 1984, 269). I will come back to several of these passages later on. But note already here that they are all from the 1890s, shortly after the publication of Dedekind's 1888 essay.[18]

11.5 DEDEKIND'S REMARKS ABOUT FREGE—AND POSSIBLE LINES OF INFLUENCE

As I am interested in the relation between Frege and Dedekind in general, let me mention Dedekind's direct remarks about Frege too. Actually, there are only two—but noteworthy ones. To supplement them, I will also discuss a comment by Dedekind that is related to Frege in a more indirect way; and I will add some reflections, or at least speculations, on ways in which Dedekind may have influenced Frege.

I already mentioned that Frege expressed disappointment, or even despair, about the fact that his foundational works were largely ignored, especially initially. As he puts it in the Preface to *Basic Laws*, volume I, from 1893:

> One searches in vain for my *Grundlagen der Arithmetik* in the *Jahrbuch über die Fortschritte der Mathematik*. Researchers in the same area, Mr. Dedekind, Mr. Otto Stolz, Mr. von Helmholtz, seem not to be acquainted with my works. Kronecker does not mention them in his essays on the concept of number either. (Frege, 1893, xi, fn. 1)

There is some justice to Frege's complaint, no doubt. However, it is noteworthy, and somewhat ironic, that in the same year, 1893, the second edition of Dedekind's *The Nature and Meaning of Numbers* appeared in print, and in its Preface we can read the following:

> About a year after the publication of my memoir I became acquainted with G. Frege's *Grundlagen der Arithmetik*, which had already appeared in the year 1884. However different the view of the essence of number adopted in that work is from my own, it contains, particularly from §79 on, points of very close contact with my paper, especially with my definition (44) [of the notion of chain]. The agreement, to be sure, is not easy to discover on account of the different form of expression; but the positiveness with which the author speaks of the logical inference from n to $n + 1$ (page 93, below) shows plainly that here he stands upon the same ground with me.
> (Dedekind, 1963, 42–3)

Not only had Dedekind discovered Frege's works by then, as he notes; he now brought it to the attention of others. He also highlights the close connection between their respective treatments of mathematical induction. Another

[18] Yet another text from the same period is Frege's 1895 review of Ernst Schröder's *Vorlesungen über die Algebra der Logik*. But interestingly, Dedekind is not mentioned in it (Frege, 1984, 210–28), despite the fact that Schröder discusses ideas from Dedekind's work sympathetically. Similarly for Frege's reviews of writings by Husserl (published in 1894) and by Peano (published in 1897); see Frege (1984).

point evident from this passage is that Dedekind came up with his main ideas independently of Frege.

Both Dedekind's belated discovery of Frege's writings, in 1889, and his sense that their logicist treatments of mathematical induction are closely related come up again in the only other Dedekindian remark on Frege of which I am aware. Namely, in a letter to the teacher Hans Keferstein, from 1890, he writes:

> Frege's *Begriffsschrift* and *Grundlagen der Arithmetik* came into my possession for the first time for a brief period last summer (1889), and I noted with pleasure that his way of defining the non-immediate succession of an element upon another in a sequence agrees in *essence* with my notion of chain (articles 37 and 44); only, one must not be put off by his somewhat inconvenient terminology. (Dedekind, 1890, 101)

Actually, we know by now—from early drafts of *The Nature and Meaning of Numbers* in Dedekind's *Nachlass*—that most of that essay's content was already in place in the early 1870s, thus well before the publication of Frege's *Begriffsschrift* (1879).[19] This confirms the independence of Dedekind's work further.

On the other hand, earlier in the 1880s Dedekind encountered ideas very close to Frege's through another route. The relevant evidence occurs in a letter, from 1888, to the mathematician Heinrich Weber, with whom Dedekind had collaborated in the 1880s. In that letter he responds to Weber's suggestion to construct the natural numbers as equivalence classes of classes. (It seems that this idea was in the air, although Frege was the first to work it out systematically.)[20] He does not dismiss Weber's approach. On the contrary, in a passage typical for Dedekind's intellectual tolerance and curiosity he writes: "I would recommend very much that you follow, at some point, this line of thought all the way through" (Dedekind, 1932, 489, my translation). Evidently Dedekind considered Weber's cardinal conception and his own ordinal conception as both worthy of further exploration—and as not necessarily standing in conflict.

If Frege's works did not influence Dedekind's foundational writings, what about the reverse: Was Frege possibly influenced by Dedekind? I am not aware of any Fregean comments on Dedekind's 1872 essay, *Continuity and Irrational Numbers*, until volume II of *Basic Laws* (1903), although he could obviously have read it much earlier. More generally, Dedekind is mentioned neither in *Begriffsschrift* (1879) nor in *Foundations of Arithmetic* (1884), which suggests that these texts were written independently of him. What about *Basic Laws* (1893/1903) however? Let me make three observations in this connection that are not original to me but perhaps not widely known. First, there is evidence that, shortly after the publication of *The Nature and Meaning of Numbers*, Frege not only read it carefully, but taught a seminar on that essay—in the

[19] Cf. again Sieg and Schlimm (2005), also for further details concerning those drafts. (I am indebted to Ansten Klev for reminding me about the quoted passage from Dedekind (1890).)

[20] Dedekind himself considers equivalence classes of classes under the relation of 1–1 mappability, but not as candidates for the natural numbers; cf. (Dedekind, 1888b, def. 34).

fall semester of 1889–90.[21] (Frege seldom taught such classes, so that this fact is noteworthy in itself.) Second, there are some closely related themes in the treatment of arithmetic in Dedekind's 1888 essay and in Frege's *Basic Laws*, volume I, even if Frege does not highlight that fact.[22] And third, while classes, or Fregean "extensions of concepts", are used only tentatively in *Foundations*, in volume I of *Basic Laws* they have become central; and of course, in Dedekind's essay his version of classes ("systems") play a prominent role.

Expanding on the third point, might Frege have been convinced by Dedekind's essay that the use of classes was definitely the way to go?[23] It is hard to be sure, since the evidence is so sparse. Many of the criticisms of Dedekind in volume I of *Basic Laws* do confirm, however, that the publication of his theory of "systems" provided a main occasion for Frege to think through, very carefully, what a theory of classes should look like. Recall also that the earliest references to Dedekind in Frege's writings, most of which concern his treatment of "systems", occur in 1890–91, just after Frege had taught his seminar on *The Nature and Meaning of Numbers.* Whether or to what degree there was a more positive influence on Frege—perhaps also concerning specifics of his treatment of arithmetic—Dedekind's essay was clearly on his mind during the period. But instead of speculating further about such positive influence, I will now turn to Frege's explicit criticisms of Dedekind.

11.6 FREGE'S MINOR CRITICISMS—AS WELL AS SOME ABSENT ONES

Let me start with four Fregean objections to Dedekind that come up, as a group, in the Introduction to *Basic Laws*, volume I. They all concern details of Dedekind's discussion of "systems". We already encountered the first in Frege's letter to Peano from 1891. It is that Dedekind's discussion, like remarks by other writers, obscures the difference between the element and the subset relation, or between the relation of an object falling under a concept and that of a concept being subordinated to another concept. This point is repeated in *Basic Laws* (Frege, 2013, 2). A second objection concerns Dedekind's somewhat unclear, or unsettled, remarks about singleton sets. Namely, he seems to suggest that we identify a set $\{a\}$ with its element a, a move Frege finds very problematic (*ibid.*).[24] Dedekind also resists the introduction of the empty set, although he acknowledges that it could be added for other purposes. For Frege, that constitutes a third weakness (*ibid.*). Fourth, what these

[21] For basic information about this seminar, cf. Frege (1969, 340, 345).

[22] For these arithmetic themes, cf. Sundholm (2001), Richard Kimberly Heck's critical reaction to Sundholm in Heck (2012), and William Stirton's contribution to the present volume.

[23] This question, and a positive answer to it, is suggested in Sundholm (2001, 61).

[24] There are complications concerning the issue of singleton sets in Frege's own work; see, e.g., fn. 1 of §10 in Frege (1893, 18). Roy Cook's essay in the present volume elaborates on the significance of this topic for Frege, thus providing deeper reasons for why he was sensitive to it.

initial points suggest, if seen together, is that Dedekind thinks of "systems" as "constituted" by their elements, another aspect Frege finds objectionable (*ibid.*).

Now, it is true that Dedekind's remarks on these topics make him an easy target for attack. He clearly didn't think them through fully; or at least, he didn't write about them carefully enough.[25] Nevertheless, these first four Fregean criticisms seem to me all relatively minor, i.e., not to carry much weight in the end. After all, Dedekind explicitly endorses an extensional notion of "system", as Frege himself notes. He also does not adopt, say, a mereological view of them, as is pretty clear. Nor does he mix up the element and the subset relation in any of his theorems. Such facts seem to me more decisive than Dedekind's somewhat careless formulations. What we can see so far, then, is that Frege tends to read Dedekind uncharitably.[26]

A fifth Fregean criticism in *Basic Laws*, still concerning Dedekind's treatment of "systems", is more significant, at least in terms of its historical impact. Here Frege puts his finger on the following passage from *The Nature and Meaning of Numbers*: "It very often happens that different things $a, b, c, \ldots$ considered for whatever reason under a common point of view, are joined together in the mind, so that they are said to form a system S" (Frege, 1893, 2; cf. Dedekind, 1963, 45). This passage elicits some typical Fregean responses against psychologistic views. (He asks: In whose mind? Are they then entirely "subjective"? Etc.). And as Frege's anti-psychologist arguments are well known and widely accepted, they leave Dedekind with the stigma of holding crude, obviously problematic views. But again, I find this a superficial and often over-interpreted point. Dedekind's remark can be understood as an informal statement in a pedagogical context. (I can easily imagine a set theorist today saying something similar when giving students an initial, intuitive grasp of the notion of set.) One can also diffuse the psychologism charge against him more generally, as I would argue, although this is not the place to do so.[27]

Let me defend Dedekind against a sixth Fregean criticism in *Basic Laws* as well, now from volume II (cf. Frege, 1903, 140–9). This is the objection that, like other mathematicians at the time (such as Hankel and Stolz), Dedekind simply "postulates" the existence of mathematical objects or systems of objects, or put differently, that he avails himself of a kind of "creative definition" that is not justified by him and indefensible in the end. (Obviously there is a connection to Russell on "theft" versus "honest toil".[28]) Dedekind does talk

[25] Dedekind was partly aware of this weakness; cf. Dedekind (1888a) for the issue of $\{a\} = a$. (The relevant part of this letter is not reprinted and translated in Ewald (1996).)

[26] As illustrated in Tait (1997), this is part of a more widespread tendency in Frege. (I take Tait's essay to be a helpful corrective to Dummett (1991), even though it is too negative about Frege overall.)

[27] See Reck (2003) and Yap (2017) for more. For the historical impact of this kind of criticism, cf. also Reck (2013).

[28] Reck (2013) contains relevant references, as well as a history of this charge against Dedekind.

about "free creation", as we saw. But I submit that it is misleading, and unjustified in itself, to lump him with the other writers mentioned in this context. Note, specifically, that Dedekind's introduction of both the real numbers, in the 1872 essay, and the natural numbers, in 1888, is preceded by an existence proof, indeed by the construction of relevant entities. For the reals, this is the construction of the system of Dedekind cuts; for the natural numbers, it is the construction of a simple infinity. Hence there is at least some "honest toil" on Dedekind's side. However, I realize that more will have to be said to absolve him completely from "theft" in this context (see below).

To close the present section, I want to mention several possible criticisms of Dedekind that—contrary to what one might expect—do not occur in *Basic Laws*, at least not directly. First (and *pace* Boolos), Frege does not attack Dedekind's proof procedure in Theorem 66, involving the notion of "thought", as psychologistic. In fact, and as already mentioned above, in his unpublished essay 'Logic' (1897) he attributes an objective notion of "thought" to Dedekind that is similar to his own. Frege also (*pace* Dummett) does not attack Dedekind's notion of "abstraction" as psychologistic. It is only Dedekind's remarks about "systems" that are criticized as such. (Frege does object to Dedekind's appeal to abstraction, to be sure, but along different lines; more on that later.) Furthermore, Frege (unlike Russell and others) does not call into question Dedekind's structuralist view of mathematical objects, at least not explicitly. Finally, not even Dedekind's ordinal conception of the natural numbers comes in for direct attack (as a view in itself), although admittedly Frege insists on a different analysis of the notion of cardinal number.

Suppose it is granted, at least temporarily or for the sake of the argument, that my defenses of Dedekind in this section have some force. Are there other Fregean criticisms that are more central and lasting? This is the topic of the next section.

11.7 FREGE'S MORE CENTRAL AND LASTING CRITICISMS

I now want to discuss five additional, more important, and more lasting criticisms of Dedekind that can be found in Frege's writings as well, especially in *Basic Laws*. These criticisms are not unrelated; indeed, they build on each other. The first two come up, together, in the following passage from the Preface to volume I:

> [N]owhere in [Dedekind's] essay do we find a list of the logical or other laws he takes as basic; and even if it were there, one would have no chance to verify whether in fact no other laws were used, since, for this, the proofs would have to be not merely indicated but carried out gaplessly. (Frege, 1893, VIII)

Let me begin by focusing on the second half of this passage. Why exactly is it a problem, according to Frege, that Dedekind's proofs are "merely indicated" and not "carried out gaplessly"? This is so because in the present context it matters for any theorem "how its proof is conducted, on what foundations it rests", as Frege goes on to say (*ibid.*). His own Begriffsschrift is, of course, designed specifically to formulate "gapless" proofs, while Dedekind has nothing comparable to offer. Frege's first major and remaining criticism of Dedekind concerns this lack.

As we are quite familiar nowadays with Frege's exacting standards for proofs, let me make a few ameliorating remarks in this context as well, without denying the aptness of this criticism in the end. Namely, in most cases where Dedekind only sketches arguments in *The Nature and Meaning of Numbers* the missing steps are not hard to fill in. Also, if one applies the proof standards common in mathematical practice, the way in which he presents things is actually detailed, relatively explicit, and quite elegant. On the other hand, Dedekind's proofs do have important "gaps", in the sense of going through only if one assumes principles of which he was most likely not aware. Thus, Ernst Zermelo observed some time ago that Dedekind's treatment of the infinite relies on implicit applications of the Axiom of Choice. More recently, it has been pointed out that, if reconstructed set-theoretically, Dedekind's procedure also involves implicit uses of the Axiom of Replacement.[29]

Frege's point that, in the absence of proofs spelled out "gaplessly", significant presuppositions may sneak in is clearly justified, as Dedekind's case illustrates well. A related but more general point is that it is actually not clear, especially from today's perspective, how to think about the general framework within which Dedekind works. In other words, even if one is willing to fill in certain gaps for him (by using set theory or higher-order logic), there is a question about how best to do so. But such considerations already point towards Frege's next major point.

Frege's second lasting criticism of Dedekind is contained early in the passage above: "[N]owhere in [his] essay do we find a list of the logical or other laws he takes as basic." So as to understand the force of this remark better, it helps to divide the missing Dedekindian "laws" into two parts (as Frege does not). On the one hand, there should be basic principles for the "constructions" used by Dedekind; on the other hand, there should be parallel principles for "Dedekind abstraction". Concerning the former, I already mentioned that, at certain crucial points, he does not just "postulate" the existence of mathematical objects, but provides relevant constructions (of the system of cuts on the rational numbers, in the case of the reals, and of a simple infinity, for the natural numbers). Dedekind does not identify the real numbers or the natural numbers with the constructed entities. Still, they, or their constructions, are crucial for him. But if so, should he not make explicit the principles underlying them? Frege is clearly right that he should, I think.

Let me expand on this point even further. In connection with his constructions, Dedekind is often assumed to rely implicitly on a "naïve" comprehension principle, where for every property, concept, or open formula a corresponding "system" is taken to exist. But it is not entirely clear that he

[29] For the axiom of choice, cf. Ferreirós (1999, 237); for replacement, cf. Kanamori (2012).

does (which only confirms Frege's criticism). It may be that the principle intended by Dedekind, or its most adequate reconstruction, is different.[30] Dedekind also does not reduce functions to sets, as is usual today. He would thus need separate existence assumptions for them. (Or perhaps sets are reduced to functions in the end?) In any case, something has to be added or changed here, already because of Russell's antinomy. Note, furthermore, that we are dealing with existence assumptions for objects that are "inflationary" in this context, i.e., with constructions that lead to strong cardinality requirements for the basic domain. (Simple infinities are countably infinite; the system of Dedekind cuts is uncountable.) This aspect requires that we are especially careful.

The situation for "Dedekind abstraction" differs significantly. Crucial here are the basic, but also missing, principles needed to underwrite Dedekind's "creation" of the natural and real numbers. In this context we start with an already constructed system, such as the system of cuts on the rational numbers, so as to introduce an isomorphic copy of it, one whose elements only have structural properties; similarly for the natural numbers. Why does Dedekind add the latter step, i.e., why does he not work directly with, say, the system of cuts? This has to do with "purity", i.e., with the fact that the cuts have "foreign" properties, ones we do not want to ascribe to the real numbers.[31] The main point for present purposes is this: Like the "construction" side of Dedekind's procedure, its "abstraction" side would seem to require basic laws, indeed ones that are interestingly different. While the laws for construction will be "inflationary" with respect to cardinality, as already noted, those for abstraction need not be. On the other hand, novel questions about identity may arise in connection with the entities resulting from "Dedekind abstraction".

Now consider the following: Why in particular, besides the general issues already raised, does Frege point to the lack of explicit principles in Dedekind's foundational work? There are three related reasons, resulting in three additional criticisms of Dedekind. The first of these emerges if we return to Frege's critique of various views about the real numbers in volume II of *Basic Laws*. As Frege notes specifically, the thinkers who appeal to "creation" in that context (Hankel, Stolz, Cantor, and Dedekind) have neglected to inquire into the limits of that procedure.[32] Surely consistency is one such limit. But in Frege's eyes, a more general, systematic investigation of these limits is called for. Yet how could one even start with the latter except by making explicit, and by then scrutinizing carefully, the underlying laws? (Frege's elaboration of his own foundational system, including Basic Law V, is exactly meant to

[30] Dedekind might rely on a "dichotomy" conception of classes instead; cf. Ferreirós (forthcoming).

[31] For this and similar points concerning Dedekind's structuralism, cf. Reck (2003). For a related discussion, see also Michael Hallett's contribution to this volume.

[32] For more on Frege's criticisms of Hankel, Stolz, Cantor, Dedekind, etc., as focused on the issue of "creation", cf. again the contribution by Michael Hallett to this volume.

ground such an investigation.) His third lasting charge against Dedekind is, thus, that he does not provide anything analogous.

It seems to me that in one sense Frege is not fair to Dedekind here, while in another sense he is. Frege is unfair insofar as he simply lumps Dedekind with other writers who are arguably less sensitive to the point at issue. Note here, in addition to my earlier defenses of Dedekind, that many of the remarks Frege finds fault with in the relevant quotations in *Basic Laws* come from Stolz. Moreover, Dedekind himself does emphasize the importance of establishing consistency, most explicitly in his correspondence.[33] Nevertheless, Frege's charge is fair insofar as Dedekind does, once again, not provide the necessary background for a relevant investigation, by not formulating basic principles. Indeed, if my suggestions above are on the right track, he would have had to provide two different kinds of principles (for "construction" and for "abstraction", respectively), bringing with them subtly different worries about consistency, different kinds of limits, and so on. A systematic exploration of such differences would be called for as well.

Turning to my fourth remaining criticism of Dedekind, Frege is concerned about the following issue too: Not only should we not use approaches that proceed piecemeal when introducing mathematical entities, as some writers do, since that is less than systematic (and, among others, increases the danger of inconsistency). We should also work with as few basic principles as possible, and ideally, just with one, like his own Basic Law V. Why might that be important? Considerations of simplicity, economy, and similar factors are relevant here, so essentially pragmatic aspects. But these do not exhaust Frege's concerns, if they count at all. More important for him are epistemological issues, and in particular, the question of what ensures our cognitive access to mathematical objects. Here is how Frege makes the crucial connection in volume II of *Basic Laws*:

> If there are logical objects at all—and the objects of arithmetic are such—then there must also be a means to grasp them, to recognize them. The basic law of logic which permits the transformation of the generality of an equality into an equality serves for this purpose. Without such a means, a scientific foundation of arithmetic would be impossible. (Frege, 1903, 149)

This passage was written before Russell told Frege about his antinomy. After finding out about it, Frege reiterates the basic point in his Afterword:

> [I] do not see how arithmetic can be founded scientifically, how the numbers can be apprehended as logical objects and brought under consideration, if it is not—at least conditionally—permissible to pass from a concept to its extension. (Frege, 1903, 253)

The main point here is that, just like Basic Law V was supposed to play the decisive role in Frege's system, a corresponding foundational principle, or a

[33] cf. Dedekind's well-known letter to Keferstein (Dedekind, 1890) among others.

few such principles, would have to take its place in another system. But again, that is just what is missing in Dedekind.

One final, fifth criticisms can be added. Note that for Frege, in both passages just quoted, it is a matter of apprehending numbers "as logical objects". The issue here is this: It is only once we have made explicit our basic principles that we can inquire whether they are "logical" principles; it is only thus that we can determine whether the corresponding objects are "logical"; and it is only along such lines that we can check whether the logicist project—Frege's and Dedekind's—has been carried out successfully or not. Actually, this last concern is raised very early in Frege's discussion of Dedekind, in the Preface to volume I of *Basic Laws*, and in a passage quoted in part already:

> Mr. Dedekind too is of the opinion that the theory of numbers is a part of logic; but his essay barely contributes to the confirmation of this opinion since his use of the expression "system", "a thing belongs to a thing" are neither customary in logic nor reducible to something acknowledged as logical. (Frege, 1893, VIII)

It is quite surprising—and problematic in itself—that Frege appeals to what is "customary in logic" and "acknowledged as logical" in this passage, so to common practice and opinion. Surely one wants a more principled criterion for what counts as "logical", especially as a Fregean. It also seems unfair to claim that Dedekind's work "barely contributes" to a confirmation of logicism, given his many technical achievements. Still, Frege has a point, i.e., there is something crucial missing for Dedekind in this respect as well. Without knowing what his basic principles are, it is, indeed, impossible to determine whether his logicist project has succeeded or not.[34]

11.8 TOWARDS A RECONCILIATION OF FREGE AND DEDEKIND

I want to round off my discussion of Frege's various criticisms of Dedekind with some more constructive remarks. This will lead to a suggestion for how to reconcile Fregean and Dedekindian approaches, i.e., for how to see them as complementary rather than as diametrically opposed. It will also point towards a way of updating Dedekind's approach, and thus, suggest a "neo-Dedekindian" research program parallel to the familiar but more developed "neo-Fregean" program. In the previous section, I distinguished between principles for "Dedekind construction" and for "Dedekind abstraction". Both are missing in his writings, at least in an explicit, precise form, as Frege highlighted. Now, rather than taking this lack to constitute a refutation of Dedekind's approach, one can see it as providing a positive challenge. Namely, is there a way of supplementing—in a Dedekindian spirit—what he did not provide himself? And if so, what form or forms could that take?

[34] There is also a question about what exactly Dedekind means by "logic". Yet the same question arises for Frege; and as his appeal to "custom" indicates, he is not fully clear on this issue either.

Consider "Dedekind construction" first. Whether or not Dedekind himself used a naïve comprehension principle for this purpose, we definitely have to be more careful and systematic than he was. But how exactly could we proceed? Four alternatives come to mind. First, we can try to take Dedekind's appeal to "thoughts" in Theorem 66 seriously, in the following sense: We work with an intensional logic as the background theory (something in the tradition of Alonzo Church's "logic of sense", say), including corresponding existence principles. We then inherit a host of doubts associated with intensional logic, however, not only about consistency (Russell's antinomy for propositions looms large), but also about criteria of identity for thoughts (made prominent by Quine). I assume, therefore, that few philosophers of mathematics will find this first option very attractive today.[35]

A second way to go would be to embed Dedekind's procedure in axiomatic set theory, say ZFC, using its axioms as our "construction principles". Indeed, this is basically what is done in contemporary set theory. Note, incidentally, than even Dedekind's much maligned "proof" of Theorem 66 plays a role in this context, since it can be, and was, seen as an inspiration for the set-theoretic axiom of infinity.[36] A third option with respect to systematizing Dedekind's approach to "construction" would be to reconstruct it in category theory. This fits well with his move to take functions as basic, his focus on homomorphisms, his use of quotient structures, etc. And again, one can see current category theory as already providing much of what is needed. Both of these approaches are very substantive mathematically. However, in both cases the result is generally not accepted as a form of logicism.

A fourth option might look more promising with respect to the goal of providing a form of logicism in the end. It is also particular apt in the context of the present essay. Consider the use of Fregean "abstraction principles" against the background of second-order logic, as proposed by Crispin Wright, Bob Hale, and their neo-logicist co-workers.[37] The suggestion is, then: Why not employ such principles to underwrite "Dedekind construction" (but not "Dedekind abstraction")?[38] As one benefit, a neo-Dedekindian might be able to appropriate much of the technical work already done by neo-logicists. But here too, a number of problems and open questions will be inherited. Nev-

[35]Then again, the core idea of using "thoughts", "thoughts about thoughts", etc., in Dedekind's "proof" of Theorem 66 goes back far in philosophy. As often noted, Bernard Bolzano gives a similar proof earlier in the nineteenth century, independently. But the idea can be traced all the way back to Aristotle's *Metaphysics, Gamma*; cf. Klev (2018). It also retains a basic informal appeal as a relatively simple illustration of an infinite sequence of entities.

[36]Start with the empty set, i.e., let it take the place of Dedekind's "self"; then replace his successor function in terms of "thoughts" with the von Neumann successor function, where n is mapped to $n \cup \{n\}$; or alternatively, use Zermelo's successor function, where n is mapped onto $\{n\}$. Either way, the result is quite close to how Dedekind proceeded, as Zermelo was well aware.

[37]Cf. Hale and Wright (2001), Cook (2007), and Ebert and Rossberg (2016), also in terms of further references.

[38]Cf. Simons (1998) for basically this suggestion.

ertheless, and most strikingly for present purposes, this approach promises a fruitful way of combining Frege and Dedekind.

However, no matter which of these four alternatives one adopts, it will provide a Dedekindian only with half of what is needed, namely, with a way of systematizing "Dedekind construction". "Dedekind abstraction" should be seen as separate. With respect to it, additional, subtly different principles are still required. As already noted, these need not be "inflationary" in terms of cardinality; but they must underwrite the introduction of "purely structural objects". Why, again, would one want to introduce such objects, in addition to those resulting from "Dedekind construction"? For the reason indicated by him: they have no "foreign", inappropriate properties. Actually, such an approach has an additional benefit not mentioned yet. If successful, it would allow for a reconciliation of the following two claims: Yes, it is "Frege abstraction" that provides us with an analysis of the notion of cardinal number. But it is "Dedekind abstraction" that distills out the conceptual minimum required for "pure arithmetic".[39] A big remaining question is, then, whether basic principles for the latter can be formulated systematically.[40]

Let me wrap things up. What I have explored in this essay is the nature of the relationship between Frege and Dedekind, as reflected in their writings. I have done so by discussing a variety of criticisms Frege raised against Dedekind, mainly in *Basic Laws of Arithmetic*. My discussion included determining which of these should be seen as more minor and which as major and lasting. At the end, I suggested that "Frege abstraction" and "Dedekind abstraction" might be seen as complementary rather than as opposed, and thus, that a Fregean approach need not be taken to be in irreconcilable conflict with a Dedekindian approach. I did not spell out the latter in detail; much work remains if one wants to show that it is a viable option. But I hope enough has been said to make it plausible that being "pro-Dedekind" does not necessarily imply being "anti-Frege". My basic conclusion is thus the following: Despite all of Frege's criticisms of Dedekind, in *Basic Laws* and beyond, their relation should not be seen as one of unmitigated, unbridgeable opposition.

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[39] This observation goes back to Ernst Cassirer in the 1910s; cf. Reck (2013).

[40] Cf. Linnebo and Pettigrew (2014) and Reck (2018) for recent, systematic, and directly relevant proposals. I plan to address this issue further in future publications, both historically and philosophically.

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